
RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 1

Fall 2023

1. The Scales of the Universe

- (a) The Planck length, time and mass are

$$\ell_P = \sqrt{\frac{\hbar G_N}{c^3}}, \quad t_P = \sqrt{\frac{\hbar G_N}{c^5}}, \quad M_P = \sqrt{\frac{\hbar c}{G_N}}. \quad (1)$$

in SI units, they are

$$l_P = 1.62 \times 10^{-35} \text{m}, \quad t_P = 5.39 \times 10^{-44} \text{s}, \quad M_P = 2.18 \times 10^{-8} \text{kg}. \quad (2)$$

While the Planck length and time are way smaller than any intervals of length and time we are able to measure, the Planck mass is well within the scales of the Universe which we experience in our everyday life. This is a first hint of the weakness of gravity, which we will discuss more in the rest of this exercise.

- (b) To find the relation between SI units and natural units, we first compute Newton's gravitational constant in GeV . Since we found that in natural units $M_P = \frac{1}{\sqrt{G_N}}$, and mass has units of energy when $c = 1$, we expect G_N to have units of E^{-2} . We notice that we can write

$$G_N \approx 6.67 \times 10^{-11} \left(\frac{\text{kg m}^2}{\text{s}^2} \right)^{-2} \frac{\text{m}^7 \text{kg}}{\text{s}^6} \quad (3)$$

where we isolated in the brackets the units of energy in SI units. Now we use $\hbar$ and c to construct the units outside of the brackets and we convert the units inside the brackets to GeV by using

$$1 \frac{\text{kg m}^2}{\text{s}^2} = 6.24 \times 10^9 \text{GeV}. \quad (4)$$

We obtain

$$G_N \approx 6.71 \times 10^{-39} \text{GeV}^{-2} \hbar c^7. \quad (5)$$

Notice that this is still true in SI units, if we consider $\hbar$ and c dimensionful. When we go to natural units, we obtain

$$G_N \approx 6.71 \times 10^{-39} \text{GeV}^{-2}. \quad (6)$$

Now to find what is the equivalence between SI units and natural units, we can simply look at the expressions for the Planck length, time and mass in SI units and compare with our value of G_N in natural units

$$\begin{aligned} l_P &= \sqrt{G_N} \longrightarrow 1\text{m} \approx 5.06 \times 10^{15} \text{GeV}^{-1}, \\ t_P &= \sqrt{G_N} \longrightarrow 1\text{s} \approx 1.52 \times 10^{24} \text{GeV}^{-1}, \\ M_P &= \frac{1}{\sqrt{G_N}} \longrightarrow 1\text{kg} \approx 5.60 \times 10^{26} \text{GeV}. \end{aligned} \quad (7)$$

Alternative solution :

We can write

$$\begin{aligned}
1\text{m} &= 1 \left(\frac{\text{kg m}^2}{\text{s}^2} \right)^{-1} \frac{\text{kg m}^3}{\text{s}^2} = \frac{1}{6.24 \times 10^9 \text{GeV}} \frac{\hbar}{1.05 \times 10^{-34}} \frac{c}{3 \times 10^8} \approx 5.06 \times 10^{15} \text{GeV}^{-1} \\
1\text{s} &= 1 \left(\frac{\text{kg m}^2}{\text{s}^2} \right)^{-1} \frac{\text{kg m}^2}{\text{s}} = \frac{1}{6.24 \times 10^9 \text{GeV}} \frac{\hbar}{1.05 \times 10^{-34}} \approx 1.52 \times 10^{24} \text{GeV}^{-1} \\
1\text{kg} &= 1 \left(\frac{\text{kg m}^2}{\text{s}^2} \right) \frac{\text{s}^2}{\text{m}^2} = 6.24 \times 10^9 \text{GeV} \frac{(3 \times 10^8)^2}{c^2} \approx 5.6 \times 10^{26} \text{GeV}
\end{aligned} \tag{8}$$

(c) • **Atoms**

To estimate the size of an atom, let us start by writing its energy in natural units

$$E \sim \frac{p^2}{2m_e} - \frac{\alpha}{r}. \tag{9}$$

In this unit system, the de Broglie wavelength of the electron is $\lambda \sim \frac{1}{p}$. Moreover, to fit in an orbit of radius r , we need $\lambda = \frac{2\pi r}{n}$ for some integer n . Considering the ground state, and neglecting factors of 2 and π , we thus get that $p \sim \frac{1}{r}$. Substituting this in our expression for the energy, we have

$$E \sim \frac{1}{2m_e r^2} - \frac{\alpha}{r}. \tag{10}$$

Bound states sit at a minimum of the energy. This lets us estimate the size of the orbit of the electron, or in other words the size of an atom:

$$\frac{1}{2m_e r_A^3} \sim \frac{\alpha}{r_A^2} \longrightarrow r_A \sim \frac{1}{2\alpha m_e} \sim 5 \text{GeV}^{-1} \sim 1 \text{\AA}, \tag{11}$$

which is twice the Bohr radius of a Hydrogen atom. For a neutral atom with Z protons we would in first approximation get

$$r_A \sim \frac{1}{2Z\alpha m_e}, \tag{12}$$

where we used that the energy becomes approximately $E \sim \frac{Z}{2m_e r^2} - \frac{\alpha Z^2}{r}$. The **binding energy**, instead, is the energy required to strip the electron off of the atom (in other words, to counteract the attraction of the potential)

$$E_b = \frac{\alpha}{r_A} \sim 2\alpha^2 m_e \sim 10 \text{ eV}. \tag{13}$$

More in general,

$$E_b = \frac{Z^2 \alpha}{r_A} \sim 2(Z\alpha)^2 m_e \sim 2Z^2 \cdot 10 \text{ eV}. \tag{14}$$

• **Solids**

We can first estimate the **number density** n of atoms in a solid by considering the fact that one atom occupies a volume of roughly $\sim r_A^3$. That means

$$n_{\text{solids}} \sim \frac{1}{V} \sim \frac{1}{r_A^3} \sim (2\alpha m_e)^3 \sim 10^2 \text{ keV}^3. \quad (15)$$

Consider that 1 keV is exactly the energy scale of X-rays. That is why X-rays can pass through certain materials (but not all: here the atomic number becomes important again). The **mass density** will be given by

$$\rho_{\text{solids}} \sim m_p (2\alpha m_e)^3 \sim 10^8 \text{ keV}^4. \quad (16)$$

For a generic element, we would have

$$\rho_{\text{solids}} \sim 2Z m_p (2Z\alpha m_e)^3 \sim (2Z)^4 \alpha m_e^3 m_p, \quad (17)$$

where we are using that the atomic mass $A = Z + n_{\text{neutrons}} \sim 2Z$. This matches the order of magnitude for light elements like sodium (with atomic number $Z = 11$) which have density of roughly $10 \times 10^3 \text{ kg/m}^3$, and heavy metals ($A \approx 200$) have masses of order $10 \times 10^4 \text{ kg/m}^3$. **Pressure** in solids is expressed through quantities like Young's modulus or the bulk modulus. Both are quantities that describe the way in which a change in energy is related to a change in volume.

$$P \sim \frac{E}{V}. \quad (18)$$

A solid will break if

$$P_{\text{solids}} \geq \frac{E_b}{V_A} \sim 2^4 \alpha^2 m_e (2\alpha m_e)^3 \sim 10^{-23} \text{ GeV}^4. \quad (19)$$

Including the atomic number,

$$P_{\text{solids}} \sim 2^4 \frac{E_b}{V_A} \sim (Z\alpha)^2 m_e (Z\alpha m_e)^3. \quad (20)$$

These give estimates for the upper bound on what the pressure in a solid can be. You can check that this matches the order of magnitude for known values of the bulk modulus in a variety of materials. For example, in iron $P \sim 10^{11} \text{ Pa}$. The **speed of sound** in a solid is given by

$$v_s = \sqrt{\frac{P}{\rho_{\text{solids}}}} \sim \alpha \sqrt{\frac{Z m_e}{m_p}}. \quad (21)$$

• Planets

Planets are made from solids (roughly), and they hold their shape against gravity because of their internal pressure, which counterbalances the pressure created by their weight. We will estimate their **size** using this fact.

$$P_{\text{solids}} \sim P_{\text{gravitational}} \quad (22)$$

We compute the gravitational pressure from Newton's law

$$P_{\text{gravitational}} \sim \frac{E_g}{V} \sim \frac{G_N M^2}{R} \frac{1}{R^3} \sim G_N \rho_{\text{solids}}^2 R^2 \sim \frac{2^8 Z^8 m_p^2 (\alpha m_e)^6}{M_p^2} R^2, \quad (23)$$

where we used the fact that $G_N = M_P^{-2}$ in natural units, and we emphasize that M_P is Planck's mass while m_p is the mass of a proton. Now, (22) becomes

$$2^4 Z^5 \alpha^5 m_e^4 \sim \frac{Z^8 m_p^2 (\alpha m_e)^6}{M_P^2} R^2 \longrightarrow R_{\text{planet}} \sim \frac{M_P}{4 m_p m_e Z^{\frac{3}{2}} \sqrt{\alpha}}, \quad (24)$$

The most common element on Earth is iron. That makes our estimate of its size

$$R_{\text{Earth}} \sim 10^5 \text{ m}, \quad (25)$$

quite close to the correct value. Notice that our estimate for the radius of a planet can be rewritten as

$$\frac{R_{\text{planet}}}{r_A} \sim \sqrt{\frac{\alpha}{G_N m_p^2}}. \quad (26)$$

This form of this estimation manifests the fact that planets are very big with respect to atoms because, between two protons, the electromagnetic repulsion is much stronger than the gravitational attraction. We can derive the **mass** by multiplying with the density of a solid

$$M_{\text{planet}} = \rho_{\text{solids}} R_{\text{planet}}^3 \sim \frac{\alpha^{\frac{3}{2}} M_P^3}{4 m_p^2 \sqrt{Z}} \quad (27)$$

You can check that these estimates for radius and mass reproduce the orders of magnitude of rocky planets ($\sim 10^{25}$ kg) and gaseous planets ($\sim 10^{26}$ kg) by plugging in different values of Z .

- **Neutron stars**

As the name suggests, neutron stars are essentially enormous nuclei made of neutrons. Their **density** is thus simply m_p^4 and so is their **pressure**, since the binding energy of a proton is most of what makes its rest mass. To estimate their **size**, we use again the equilibrium between the pressures

$$P_{\text{n.s.}} \sim P_{\text{grav}} \longrightarrow G_N \rho_{\text{n.s.}}^2 R_{\text{n.s.}}^2 \sim m_p^4 \longrightarrow R_{\text{n.s.}} \sim \frac{M_P}{m_p^2} \sim 10^{19} \text{ GeV} \sim 10^4 \text{ m}, \quad (28)$$

which is incredibly small. You can check that their **mass** is instead of the same order of magnitude as the mass of the sun.

$$M_{\text{n.s.}} \sim \frac{M_P^3}{m_p^2} \sim 10^{57} \text{ GeV} \sim 10^{30} \text{ kg}. \quad (29)$$

- **Living beings**

In order to estimate the **size** of a living being, we need to impose that the molecular bonds along a certain cross sectional area are not broken by, for example, falling on the ground. That means we need to compare the cross sectional binding energy in a living being to the gravitational potential energy it experiences on the planet

$$\left(\frac{M_{\text{animal}}}{m_p} \right)^{\frac{2}{3}} E_b \sim M_{\text{animal}} L_{\text{animal}} g, \quad (30)$$

where $\left(\frac{M_{\text{animal}}}{m_p}\right)^{\frac{2}{3}}$ counts the number of atoms along a certain cross section of the (approximately spherically symmetric) animal. Animals are in first approximation uniform solids, $M_{\text{animal}} \sim \rho_{\text{solids}} L_{\text{animal}}^3$. Moreover, we can use the binding energy we derived before. We obtain

$$\left(\frac{\rho_{\text{solids}} L_{\text{animal}}^3}{m_p}\right)^{\frac{2}{3}} E_b \sim \rho_{\text{solids}} L_{\text{animal}}^4 \frac{GM_{\text{planet}}}{R_{\text{planet}}^2} \quad (31)$$

$$L_{\text{animal}} \sim \sqrt{\frac{M_P}{m_p}} \frac{1}{m_e} \frac{Z_{\text{animal}}^{\frac{1}{3}}}{Z_{\text{planet}}^{5/4}} \frac{1}{\alpha^{3/4}}.$$

Depending on the values we choose for Z_{planet} and Z_{animal} , we get

$$L_{\text{animal}} \sim 10 \text{ mm to } 1 \text{ m}. \quad (32)$$

The estimated **mass**, instead, is

$$M_{\text{animal}} \sim \rho_{\text{solids}} L_{\text{animal}}^3 \sim M_P \sqrt{\frac{M_P}{m_p}} \alpha^{3/4} \frac{Z_{\text{animal}}^5}{Z_{\text{planet}}^{15/4}} \quad (33)$$

which gives a range of

$$M_{\text{animal}} \sim 10^{-1} \text{ kg to } 10 \text{ kg} \quad (34)$$

Notice also that the mass of the animal is the geometric mean between a planet and a proton:

$$M_{\text{animal}}^2 \sim M_{\text{planet}} m_p \sim \frac{M_P^3}{m_p} \alpha^{3/2} \quad (35)$$

2. Index Gymnastics part 1

	Equation	Dummy	Free	# of Equations	Alternative Form
(a)	$A_\alpha B^\alpha = 5$	α		1	$A_\gamma B^\gamma = 5$
	$A^\mu = \Lambda^\mu_\nu A^\nu$	ν	μ	4	$A^\gamma = \Lambda^\gamma_\beta A^\beta$
	$T^{\alpha\mu\lambda} A_\mu C_\lambda^\gamma = D^{\alpha\gamma}$	μ, λ	α, γ	16	$T^{\beta\nu\tau} A_\nu C_\tau^\rho = D^{\beta\rho}$

- (b) δ^μ_μ is the trace of the identity matrix. In 4 dimensions, $\delta^\mu_\mu = 4$. In n dimensions, $\delta^\mu_\mu = n$. Regarding the Minkowski metric, since it is defined as $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, we need to raise one of the indices with the inverse Minkowski metric, which is also $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$

$$\eta^\mu_\nu = \eta^{\mu\alpha} \eta_{\alpha\nu} = \delta^\mu_\nu, \quad (36)$$

where we used matrix multiplication on the last equality. So, we showed that

$$\eta^\mu_\mu = \delta^\mu_\mu. \quad (37)$$

- (c) • The symmetrization can happen between indices of two different tensors that are being multiplied with each other.

$$A_{\tau(\alpha} B^\tau_{\beta)} = \frac{1}{2} (A_{\tau\alpha} B^\tau_\beta + A_{\tau\beta} B^\tau_\alpha). \quad (38)$$

- When more than two indices are involved in a symmetrization, it is useful to iterate the process:

$$\begin{aligned} A_{(\mu} B^\nu_{\tau\sigma)} &= \frac{1}{3} (A_\mu B^\nu_{(\tau\sigma)} + A_\tau B^\nu_{(\mu\sigma)} + A_\sigma B^\nu_{(\mu\tau)}) \\ &= \frac{1}{3!} (A_\mu B^\nu_{\tau\sigma} + A_\mu B^\nu_{\sigma\tau} + A_\tau B^\nu_{\mu\sigma} + A_\tau B^\nu_{\sigma\mu} + A_\sigma B^\nu_{\mu\tau} + A_\sigma B^\nu_{\tau\mu}). \end{aligned} \quad (39)$$

You can check that the result is symmetric under the exchange of any two of the three lower indices.

- To construct all the permutations of an antisymmetrized tensor with the correct signs, one needs to consider that any two terms that differ by the permutation of one single pair of indices, should have opposite signs.

$$A_{[\mu\tau\sigma]} = \frac{1}{3!} (A_{\mu\tau\sigma} - A_{\mu\sigma\tau} - A_{\tau\mu\sigma} - A_{\sigma\tau\mu} + A_{\sigma\mu\tau} + A_{\tau\sigma\mu}) \quad (40)$$

- One can have symmetrization over some indices and antisymmetrization over other indices:

$$\begin{aligned} A_{(\mu}^{[\nu} B^{\sigma]}_{\tau)} &= \frac{1}{2} (A_\mu^{[\nu} B^{\sigma]}_\tau + A_\tau^{[\nu} B^{\sigma]}_\mu) \\ &= \frac{1}{4} (A_\mu^\nu B^\sigma_\tau - A_\mu^\sigma B^\nu_\tau + A_\tau^\nu B^\sigma_\mu - A_\tau^\sigma B^\nu_\mu) \end{aligned} \quad (41)$$

3. Canonical Form of Time-, Space-, and Light-like Four-vectors

Since the spatial coordinates of a vector can be reoriented via a rotation, every vector of the form

$$V = \begin{pmatrix} V^0 \\ V^1 \\ V^2 \\ V^3 \end{pmatrix} \quad (42)$$

can be put into

$$V' = RV = \begin{pmatrix} V^0 \\ |V| \\ 0 \\ 0 \end{pmatrix} \quad (43)$$

where $|V|^2 = (V^1)^2 + (V^2)^2 + (V^3)^2$, and R is the rotation matrix acting on the spatial components. The norm is $\eta_{\mu\nu}V^\mu V^\nu = -(V^0)^2 + |V|^2$

(a) Applying a boost of rapidity η gives

$$V'' = \Lambda V' = \begin{pmatrix} \cosh \eta & -\sinh \eta & 0 & 0 \\ -\sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ |V| \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} V^0 \cosh \eta - |V| \sinh \eta \\ -V^0 \sinh \eta + |V| \cosh \eta \\ 0 \\ 0 \end{pmatrix} \quad (44)$$

To have $(V'')^1 = 0$ as required by the exercise, the rapidity must obey the equation

$$\tanh \eta = \frac{|V|}{V^0}. \quad (45)$$

Since for real arguments $\tanh$ takes values between -1 and $+1$, this is only possible if $(V^0)^2 > |V|^2$, which is equivalent to the requirement that the four-vector V is time-like:

$$\eta_{\mu\nu}V^\mu V^\nu < 0. \quad (46)$$

The temporal component then satisfies

$$(V'')^0 = V^0 \cosh \left(\tanh^{-1} \frac{|V|}{V^0} \right) - |V| \sinh \left(\tanh^{-1} \frac{|V|}{V^0} \right). \quad (47)$$

Using the hyperbolic identities suggested in the exercise, we get

$$(V'')^0 = \frac{V^0}{\sqrt{1 - \frac{|V|^2}{(V^0)^2}}} - \frac{|V| \cdot \frac{|V|}{(V^0)}}{\sqrt{1 - \frac{|V|^2}{(V^0)^2}}} = \sqrt{(V^0)^2 - |V|^2} = \sqrt{-\eta_{\mu\nu}V^\mu V^\nu} \quad (48)$$

which concludes the proof.

(b) For a space-like vector, the procedure is similar. To have $(V'')^0 = 0$, we need

$$\tanh \eta = \frac{v^0}{|v|}. \quad (49)$$

On the other hand,

$$(V'')^1 = \frac{-V^0 \cdot \frac{V^0}{|V|}}{\sqrt{1 - \frac{(V^0)^2}{|V|^2}}} + \frac{|V|}{\sqrt{1 - \frac{(V^0)^2}{|V|^2}}} = \sqrt{-(V^0)^2 + |V|^2} = \sqrt{\eta_{\mu\nu} V^\mu V^\nu}, \quad (50)$$

as we were asked to prove.

- (c) The derivation for the light-like vector involves only applying the rotation matrix. The norm $-(V^0)^2 + |V|^2 = 0$ implies $(V')^0 = (V')^1 = V^0$, which is the desired form.

4. A Primer in Variational Calculus

- (a) We want to compute the functional derivatives of the following:

$$\begin{aligned} F_1[f] &= \int f(x) dx, & F_2[f] &= \int (f(x))^p \phi(x) dx, \\ F_3[f] &= \int g[f(x)] dx, & F_4[x] &= \int \left(\frac{dx}{dt} \right)^2 dt, \end{aligned} \quad (51)$$

using the definition

$$\frac{\delta F}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{F[f(x) + \epsilon \delta(x - x_0)] - F[f(x)]}{\epsilon}. \quad (52)$$

Then we have

$$\frac{\delta F_1[f]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int (f(x) + \epsilon \delta(x - x_0) - f(x)) dx = 1, \quad (53)$$

$$\frac{\delta F_2[f]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int [(f(x) + \epsilon \delta(x - x_0))^p - (f(x))^p] \phi(x) dx = p[f(x_0)]^{p-1} \phi(x_0), \quad (54)$$

$$\frac{\delta F_3[f]}{\delta f(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int [g(f(x) + \epsilon \delta(x - x_0)) - g(f(x))] dx = g'(f(x_0)), \quad (55)$$

$$\begin{aligned} \frac{\delta F_4[x]}{\delta x(t_0)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int \left[\left(\frac{d}{dt} (x(t) + \epsilon \delta(t - t_0)) \right)^2 - \left(\frac{dx}{dt} \right)^2 \right] dt \\ &= 2 \int \frac{d}{dt} x(t) \frac{d}{dt} \delta(t - t_0) dt \end{aligned} \quad (56)$$

Integrating by parts the last line,

$$\frac{\delta F_4[x]}{\delta x(t_0)} = -2 \frac{d^2 x}{dt^2}. \quad (57)$$

- (b) We now move to functionals with vector fields

$$F_1[A^\nu] = \int A_\mu A^\mu dx, \quad F_2[A^\nu] = \int F_{\mu\nu} F^{\mu\nu} d^4 x, \quad (58)$$

where the functional derivative is defined as

$$\frac{\delta F}{\delta f^\mu(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{F[f^\nu(x) + \epsilon \delta_\mu^\nu \delta(x - x_0)] - F[f^\nu(x)]}{\epsilon}, \quad (59)$$

We therefore have

$$\begin{aligned} \frac{\delta F_1[A^\nu]}{\delta A^\rho(x_0)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int dx \left[\eta_{\mu\nu} (A^\nu(x) + \epsilon \delta_\rho^\nu \delta(x - x_0)) (A^\mu(x) + \epsilon \delta_\rho^\mu \delta(x - x_0)) \right. \\ &\quad \left. - \eta_{\mu\nu} A^\mu(x) A^\nu(x) \right] \\ &= \eta_{\mu\rho} A^\mu(x_0) + \eta_{\rho\nu} A^\nu(x_0) = 2A_\rho(x_0), \end{aligned} \quad (60)$$

Notice that the usual intuition from calculus is correct: we are deriving a square, so we expect the exponent to decrease by one and to drop in front. It is usually the case that intuition is enough to go through functional derivatives, but it is still important to here show all the rigorous steps in between, at least for once.

$$\begin{aligned} \frac{\delta F_2[A^\nu]}{\delta A^\rho(x_0)} &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int d^4x \left[\eta_{\mu\alpha} \eta_{\nu\beta} \right. \\ &\quad \times (\partial^\alpha A^\beta(x) - \partial^\beta A^\alpha(x) + \epsilon \delta_\rho^\beta \partial^\alpha \delta^{(4)}(x - x_0) - \epsilon \delta_\rho^\alpha \partial^\beta \delta^{(4)}(x - x_0)) \\ &\quad \times (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) + \epsilon \delta_\rho^\nu \partial^\mu \delta^{(4)}(x - x_0) - \epsilon \delta_\rho^\mu \partial^\nu \delta^{(4)}(x - x_0)) \\ &\quad \left. - \eta_{\mu\alpha} \eta_{\nu\beta} (\partial^\alpha A^\beta(x) - \partial^\beta A^\alpha(x)) (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) \right] \\ &= -\eta_{\mu\alpha} \eta_{\nu\beta} (\delta_\rho^\beta \partial^\alpha - \delta_\rho^\alpha \partial^\beta) (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) \\ &\quad - \eta_{\mu\alpha} \eta_{\nu\beta} (\delta_\rho^\nu \partial^\mu - \delta_\rho^\mu \partial^\nu) (\partial^\alpha A^\beta(x) - \partial^\beta A^\alpha(x)) \\ &= -\partial_\mu F_\rho^\mu + \partial_\nu F_\rho^\nu - \partial_\alpha F_\rho^\alpha + \partial_\beta F_\rho^\beta = -4\partial_\mu F_\rho^\mu. \end{aligned} \quad (61)$$

Once you got nightmares after understanding the details of this computation you can convince yourself that you can do it much faster using the physicist way

$$\begin{aligned} \delta F_2 &= \int d^4x 2F^{\mu\nu} \delta F_{\mu\nu} = 2 \int d^4x F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) \\ &= -2 \int d^4x (\partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\nu F^{\mu\nu} \delta A_\mu) = -4 \int d^4x \partial_\mu F^{\mu\nu} \delta A_\nu \\ &\implies \frac{\delta F_2}{\delta A_\nu} = -4\partial_\mu F^{\mu\nu}. \end{aligned} \quad (62)$$

(c)

$$\frac{\delta F[g^{\alpha\beta}]}{\delta g^{\mu\nu}(x_0)} = \lim_{\epsilon \rightarrow 0} \frac{F[g^{\alpha\beta}(x) + \delta_\mu^\alpha \delta_\nu^\beta \delta(x - x_0)] - F[g^{\alpha\beta}(x)]}{\epsilon}, \quad (63)$$